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*Formal Relationship*

**How lifespan and life years lost equate to unity**

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## How lifespan and life years lost equate to unity

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### Abstract

#### BACKGROUND

Life expectancy at birth ( $e_0$ ), life years lost at death ( $e^\dagger$ ), and lifetable entropy ( $\mathcal{H}$ ) are key indicators that capture average lifespan and lifespan variation. Expressions and relationships among these summary measures form the basis to analytically derive a range of formal demographic relationships, that build on each other and together help create new insights. Even though many elegant relationships are known, new ones are still to be discovered.

#### RESULTS

The sum of life expectancy and life years lost at death, scaled by the level and rate parameters  $a$  and  $b$  of the Gompertz mortality model, equals one. This plain relationship has mathematical beauty and connects key demographic measures. It directly implies further relationships, and allows connecting existing ones. It can be interpreted as a pace–shape decomposition of lifespan.

#### CONTRIBUTION

We contribute a useful relationship to complement analytical tools for studying life expectancy and lifespan variation. It can reveal macro-level regularities that may aid development of novel forecasting methods in the future. It could also support more comparative research across species by quantifying the relative impact of the environment on species' life histories. We also propose the ratio  $a/b$  as a potential metric to signal major trend changes in mortality improvements.

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## 1. Relationship

Life expectancy at birth ( $e_0$ ) is a summary measure of mortality that expresses the average number of years a newborn is expected to live assuming that current conditions will prevail over time from a period perspective (Preston, Heuveline, and Guillot 2001). From a cohort perspective it captures the average lifespan of a generation. It is defined as

$$(1) \quad e_0 = \int_0^{\omega} \ell(x) dx,$$

where  $\ell(x)$  is the lifetable survival function at age  $x$  and  $\omega$  is the highest age observed.

Life disparity ( $e^\dagger$ ) is a complementary measure that summarizes the average life years lost at the time of death (Goldman and Lord 1986; Hakkert 1987; Vaupel 1986). It indicates how strongly people differ in their ages at death and is defined as (Vaupel and Canudas-Romo 2003)

$$(2) \quad e^\dagger = - \int_0^{\omega} \ell(x) \ln \ell(x) dx.$$

It holds that  $e_0, e^\dagger \geq 0$  for all ages and for all survival patterns.

The quotient of lifespan disparity (2) and life expectancy (1) is a dimensionless indicator of relative variation in the length of life known as the lifetable entropy ( $\mathcal{H}$ ) (Leser 1955; Demetrius 1974; Keyfitz 1977).

Life expectancy results from the cumulative experience of the risk of death over age, as captured by the age pattern of mortality. Although mortality over age depends on a multitude of interacting causes both within and outside the body, theory proposed by Gompertz suggests that mortality over adult ages is well described by a simple exponential pattern (Gompertz 1825). This Gompertz mortality model is given by

$$\mu(x) = ae^{bx}.$$

Parameter  $a > 0$  specifies the initial level of mortality  $\mu(0)$  at age zero. Parameter  $b > 0$  captures the rate at which the force of mortality increases over age  $x \geq 0$ , typically referred to as the rate of aging.<sup>3</sup>

The exponential model by Gompertz is generally known to capture not just the pattern of mortality between ages 30 and 90 in human populations but also adult mortal-

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<sup>3</sup> Parameter  $b$  can be equal to or even smaller than zero for some species other than mammals (Reinke et al. 2022; da Silva et al. 2022; Baudisch Forthcoming)

ity patterns across mammals (Promislow 1991) and many other species reasonably well (Finch, Pike, and Witten 1990). And although other parametric mortality models, such as the Gompertz Makeham model (Makeham 1860) or the Gamma-Gompertz model (Manton, Stallard, and Vaupel 1986), are known to provide a better fit to the human adult mortality pattern, an exponential increase is a central component of those models. Hence, it appears to capture something fundamental about the underlying aging process.

Here we show that for the case of Gompertz mortality, life expectancy at birth (1) and life disparity (2) are connected via the plain relationship

$$(3) \quad a e_0 + b e^\dagger = 1.$$

Lifespan and life years lost, scaled by the level of mortality and the rate of aging, sum to unity. This simple connection between lifespan and life years lost at death was unknown, yet it is not new. This relationship is a special case of a previously derived formulation for the entropy of the Gompertz-Makeham model as a function of its model parameters, life expectancy, and the crude death rate (Wrycza 2014).

## 2. Proof

Wrycza (2014) starts his proof by deriving the cumulative hazard  $H(x)$  for the case of the Gompertz-Makeham mortality model. Although quite similar, here we provide an instructive alternative proof with a different starting point.

Life expectancy (1) and life disparity (2) sum to

$$(4) \quad e_0 + e^\dagger = \int_0^\omega \ell(x) dx - \int_0^\omega \ell(x) \ln \ell(x) dx.$$

With survival  $\ell(x)$  relating to the underlying cumulative hazard function  $H(x)$  until age  $x$  via  $\ell(x) = \exp[-H(x)]$ , Equation (4) changes to

$$(5) \quad e_0 + e^\dagger = \int_0^\omega \ell(x) dx + \int_0^\omega \ell(x) H(x) dx.$$

The cumulative hazard from age 0 (birth or some other chosen initial age) to age  $x$  is defined as

$$H(x) = \int_0^x \mu(a) da .$$

For a Gompertz mortality pattern, the cumulative hazard integrates an exponential function. Hence, it holds that

$$H(x) = \frac{a}{b} (e^{bx} - 1) = \frac{\mu(x)}{b} - \frac{a}{b},$$

which is a special case of Wryzca's derivation for  $c = 0$ . Together with (1), this simplifies Equation (5) to

$$(6) \quad e_0 + e^\dagger = e_0 + \frac{1}{b} \int_0^\omega \ell(x)\mu(x) dx - \frac{a}{b} \int_0^\omega \ell(x) dx .$$

Since the product  $\ell(x)\mu(x)$  gives the distribution function of deaths,  $d(x)$ , and its sum over age is one,  $\int_0^\omega d(x)dx = 1$ , Equation (6) reduces to

$$(7) \quad e^\dagger = \frac{1}{b} - \frac{a}{b} e_0 .$$

Multiplying by  $b$  and adding  $a e_0$  proves equality (3). □

### 3. Related results

**Related relationships.** Wryzca and Baudisch (2012) show that

$$(8) \quad \frac{de_0}{da} = -\frac{e^\dagger}{a}$$

and

$$(9) \quad \frac{de_0}{db} = \frac{1}{b}(e^\dagger - e_0),$$

in line with equations (2) and (3) of Wrycza (2014) for  $c = 0$ . These relationships imply that life expectancy declines as parameters  $a$  or  $b$  increase. This is true both for positive or negative aging rates, as for  $b > 0$  it holds that  $e^\dagger < e_0$ , and for  $b < 0$  it holds that  $e^\dagger > e_0$ .

Similar derivatives can be calculated for  $e^\dagger$  (derivations in the Appendix) based on (7), a reshuffled version of (3), which gives

$$(10) \quad \frac{de^\dagger}{da} = \frac{de_0}{db}$$

and

$$(11) \quad \frac{de^\dagger}{db} = \frac{a}{b} \left( \frac{de_0}{da} - \frac{de_0}{db} \right).$$

Elegant relationships also hold for the relative changes in life expectancy and lifespan disparity with respect to the mortality parameters (shown in the Appendix),

$$(12) \quad -a \frac{\frac{de_0}{da}}{e_0} - b \frac{\frac{de_0}{db}}{e_0} = 1$$

and

$$(13) \quad -a \frac{\frac{de^\dagger}{da}}{e^\dagger} - b \frac{\frac{de^\dagger}{db}}{e^\dagger} = 1.$$

These relationships capture how relative changes of life expectancy and lifespan disparity with respect to  $a$  and  $b$  weigh the contributions of these parameters to the total change. They are elegant because of their symmetry within and among each other and with Equation (3). That these relationships sum to one is useful because this can be interpreted as a weighting function that reveals the comparative importance of the level of mortality and the change in mortality over age in driving life expectancy and lifespan disparity. The negative sign can be understood by noting that life expectancy declines with increasing level and rate parameters.

Another way of using these equations is to express life expectancy and lifespan disparity in terms of changes in these variables themselves, but not including the other.

Mildly reshuffling the two above expressions gives

$$(14) \quad e_0 = - \left( a \frac{de_0}{da} + b \frac{de_0}{db} \right)$$

and

$$(15) \quad e^\dagger = - \left( a \frac{de^\dagger}{da} + b \frac{de^\dagger}{db} \right).$$

Different to the relationships above, here life expectancy depends only on sensitivities of life expectancy but not on the sensitivities of lifespan disparity. Similarly, life disparity depends only on sensitivities of life disparity but not on the sensitivities of life expectancy. Such separation can be helpful in formal derivations.

Relationships for the lifetable entropy  $\mathcal{H} = e^\dagger/e_0$  as the ratio of life expectancy and life disparity follow accordingly. Dividing the reshuffled version of our basic relationship (7) by  $e_0$  yields

$$(16) \quad \mathcal{H} = \frac{1}{b} \left( \frac{1}{e_0} - a \right),$$

an expression for lifetable entropy as an explicit function of the Gompertz parameters and life expectancy. This is again a special case of Wrycza (2014), Equation (1).

This expression can be used to derive changes in lifetable entropy with respect to parameters  $a$  and  $b$  (see Appendix) as

$$(17) \quad \frac{d\mathcal{H}}{da} = \frac{1}{b} \left( \frac{\mathcal{H}}{ae_0} - 1 \right)$$

and

$$(18) \quad \frac{d\mathcal{H}}{db} = -\frac{a}{b^2} \left( \frac{\mathcal{H}}{ae_0} - 1 \right),$$

which implies that

$$(19) \quad \frac{d\mathcal{H}}{db} = -\frac{a}{b} \frac{d\mathcal{H}}{da}.$$



Multiplying this equation by  $b$  reveals a relationship among the sensitivities of the life expectancy. If mortality follows a Gompertz pattern, then the absolute change in life expectancy with respect to the level of mortality equals the absolute change with respect to the rate of change in mortality, respectively scaled by level and rate of mortality. It follows that the sum of absolute change in the life expectancy weighted respectively by level and rate parameter of Gompertz mortality equals zero:

$$(20) \quad a \frac{d\mathcal{H}}{da} + b \frac{d\mathcal{H}}{db} = 0.$$

A similar observation is true for relative changes of life expectancy with respect to mortality parameters, where

$$(21) \quad a \frac{\frac{d\mathcal{H}}{da}}{\mathcal{H}} + b \frac{\frac{d\mathcal{H}}{db}}{\mathcal{H}} = 0.$$

These equations have a symmetry that is similar to relationships (3), (12), and (13). Different from the previous equations, however, the relationships here sum to zero. This is useful because it can be interpreted as a budget constraint between the level and rate parameter of mortality, such that any gains or losses of change in one must be balanced by gains or losses in the other.

**Alternative formulations.** For formal analysis, it can help to express sensitivities of life expectancy as a function of life expectancy alone, which holds analogously for life expectancy disparity and the life expectancy. Using the fundamental relationship (3) by expressing  $e^\dagger$  as a function of  $e_0$ , as in (7), or the other way around as appropriate, and inserting respectively for  $e_0$  or  $e^\dagger$  into the original sensitivity Equations (8) and (10) yields sets of alternative, linear relationships.

Sensitivities with respect to the level of mortality follow the positive linear relationships

$$(22) \quad \frac{de_0}{da} = -\frac{1}{ab} + \frac{1}{b} e_0$$

and

$$(23) \quad \frac{de^\dagger}{da} = -\frac{1}{ab} + \frac{a+b}{ab} e^\dagger,$$

with negative intercepts given by the inverse of the product of level and rate parameter  $a$  and  $b$ .

Sensitivities with respect to the rate of mortality follow negative linear relationships

$$(24) \quad \frac{de_0}{db} = \frac{1}{b} - \frac{a+b}{b} e_0$$

and

$$(25) \quad \frac{de^\dagger}{db} = \frac{1}{b^2} - \frac{a+2b}{b^2} e^\dagger,$$

with positive intercepts given by inverse functions of the rate parameter  $b$ .

Analogously, sensitivities of entropy can be found. They turn out to follow the quadratic relationships

$$(26) \quad \frac{d\mathcal{H}}{da} = \frac{1}{a} \left( \mathcal{H}^2 + \frac{a}{b} \mathcal{H} - \frac{a}{b} \right)$$

and

$$(27) \quad \frac{d\mathcal{H}}{db} = -\frac{1}{b} \left( \mathcal{H}^2 + \frac{a}{b} \mathcal{H} - \frac{a}{b} \right),$$

where change in  $a$  and  $b$  are given by the same expressions, just differently scaled by the inverse of the respective parameter with opposite signs, in agreement with relationship (19). The ratio  $a/b$  characterizes the quadratic equation of entropy and captures the relative magnitude of the level and rate parameter of mortality. Its interpretation is discussed below.

**Connection to previous results.** The main relationship derived in (3) links the level and rate parameters of the Gompertz mortality model with life expectancy and life disparity. It readily connects with previous results of mathematical demography.

Vaupel (1986) shows that for Gompertz mortality, the life table entropy  $\mathcal{H} = e^\dagger/e_0$  can be approximated by  $\mathcal{H} \approx 1/be_0$ . Similarly, our relationship implies that for populations with negligible baseline levels of mortality,  $a \approx 0$ , life disparity is the reciprocal of the rate of aging,  $b$ , as  $e^\dagger \approx 1/b$ .

Conversely, for populations with negligible increase in mortality over age,  $b \approx 0$ , the relationship converges to the case of constant mortality, where life expectancy equals the inverse of the hazard, here  $e_0 \approx 1/a$ .

Equation (7) allows expressing  $e^\dagger$  as a linear function of  $e_0$  with intercept  $1/b$  and slope  $-a/b$ . This is a special case of the more general case of the Gompertz–Makeham mortality model (Wrycza 2014).

Dividing the relationship in Equation (3) by  $e_0$  leads to the result that  $a + b\mathcal{H} = 1/e_0$ , or equivalently

$$(28) \quad \bar{\mu} = a + b\mathcal{H}.$$

Equation (28) is similar to Equation (1) in Wrycza (2014) for  $c = 0$ . It decomposes the average, or crude, death rate in a stationary population  $\bar{\mu}$  into the sum of the baseline mortality level and the lifetable entropy weighted by the rate of aging.

## 4. Applications

The relationships proposed in this paper contribute tools to explore and compare mortality patterns across populations. This applies especially also for analysis of human and non-human populations over time and across different environments within the pace–shape framework (Baudisch 2011).

Motivated by theoretical insights from evolutionary demographic models on how optimal resource allocation determines patterns of birth and death over age (Baudisch 2008), this framework distinguishes how long organisms live (the pace of mortality) from how mortality changes over age (the shape of mortality) to characterize aging (Baudisch 2011). The pace dimension captures lifespan and can be measured by life expectancy (Wrycza and Baudisch 2014). The shape dimension captures how relatively strongly the age pattern of mortality increases or decreases over (adult) ages, including the possibility that mortality remains constant with age. Measures of shape turn out to be equivalent to prominent measures of relative spread – for example, the Gini coefficient, the coefficient of variation, and lifetable entropy (Wrycza, Missov, and Baudisch 2015).

Within the pace–shape framework, the relationships presented in this paper are particularly useful. They can aid analysis and are intuitively appealing and interpretable. Parameter  $a$  determines the level of mortality, and hence the pace of mortality. Parameter  $b$  determines how fast mortality changes with age, and hence relates to the shape of mortality. In (3), parameter  $a$  scales life expectancy  $e_0$ , a measure of pace, while the rate parameter  $b$  scales life disparity  $e^\dagger$ , a key component of the lifetable entropy  $\mathcal{H} = e^\dagger/e_0$ , a measure of shape.<sup>4</sup>

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<sup>4</sup> It is important to note that the rate of mortality  $b$  in itself is not a shape measure. As a rate, it depends on units of time and thereby includes the signal of pace. Thus, dividing  $e^\dagger$  by  $e_0$  to get lifetable entropy as a measure of shape can be interpreted as removing the signal of pace from the processes that determine the patterns of death.

**Pace- versus shape-driven populations.** From a pace versus shape perspective, relationship (3) decomposes the impact of pace versus shape on lifespan. The first term captures the relative importance of the level of mortality (pace), and the second term captures the relative importance of how mortality changes with age (shape), which both sum to one. As above, similar interpretations hold for the other relationships, such as the weighting functions (3), (12), and (13) and the budget constraints (20) and (21), which constrain the dynamics of mortality over time in pace–shape space (with pace on the x-axis and shape on the y-axis).

A large pace component in (3) implies that life expectancy mainly hinges on environmental conditions and less so on mortality differences across the age range. By contrast, a large shape component implies that life expectancy mainly results from aging processes, whereas environmental conditions are relatively less important for how long individuals live on average.

Figure 1 illustrates these dynamics fitting Gompertz models between ages 35 and 90 by sex to 15 countries for which data is available at least from 1922 from the Human Mortality Database (HMD 2023; Barbieri et al. 2015). These included Australia, Switzerland, Denmark, Finland, Iceland, Sweden, Norway, Netherlands, Italy, England and Wales, Northern Ireland, Scotland, France, Canada, and Spain. Sensitivity analyses were carried out by fitting the models to ages between 30 and 90, and results were very similar. To fit the models, we used the Gompertz specification in the *MortalityLaws* package in R (Pascariu 2019), from which we retrieved the Gompertz parameters. To calculate life expectancy from these parameters, we used the formula given by Missov and Lenart (2013) and the formulation by Wrycza (2014).<sup>5</sup>

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<sup>5</sup> Replication materials are available here [https://github.com/CPop-SDU/Baudisch-Aburto\\_2023\\_Demographic\\_Research](https://github.com/CPop-SDU/Baudisch-Aburto_2023_Demographic_Research).

**Figure 1: Decomposition into pace (left) and shape (right) contribution to determine lifespan and lifespan variation for selected countries**

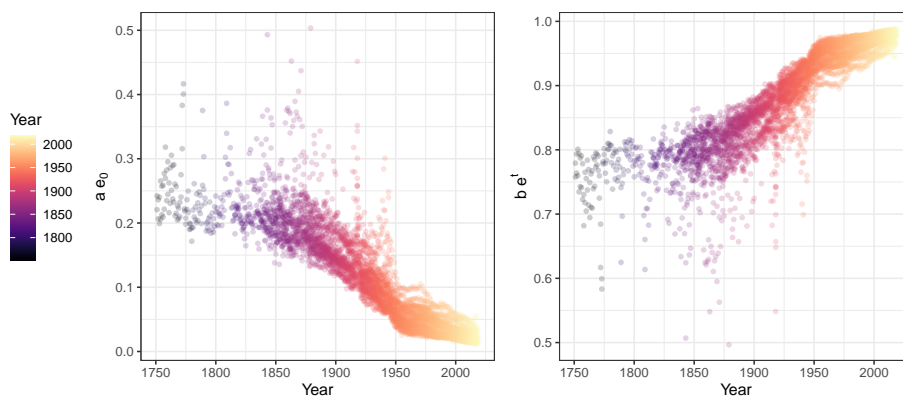


Figure 1 reveals that back to the 18<sup>th</sup> century, for human populations it was never just the overall level of environmental mortality that determined how long people lived and how much they differed in their ages at death. The fact that people grow old at a certain rate always influenced life expectancy and its disparity to at least about 75%. Nowadays, the level of mortality is so low that more than 95% of our survival prospects hinge on the rate of aging. Notably, over historical time, there are some countries where conditions have been so challenging that the overall level of death has played a close to or equal role as the age differences in mortality in determining lifespan.

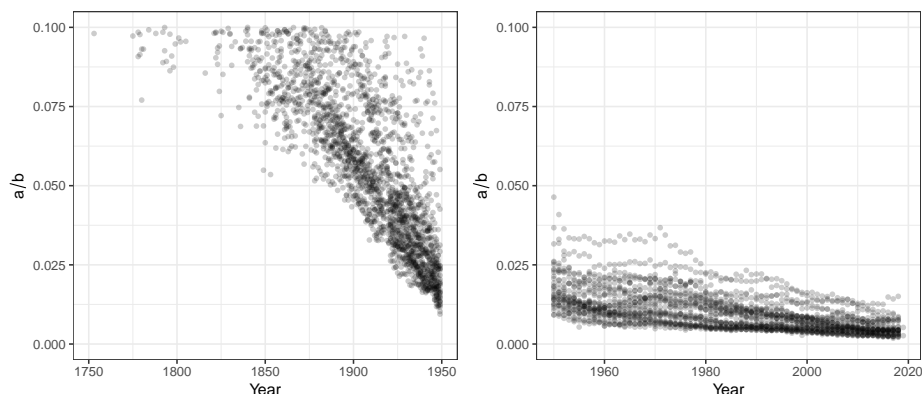
It would be interesting to compare these values to other species in changing environments. We hypothesize that in nature for nonhuman populations, the pace component would be considerably larger than the shape component, in contrast to what we observe for humans. We propose that our relationship could be used to quantify the relative impact of the environment on species' life histories in a comparative framework.

Another way of capturing the relative role of level (pace) and change (shape) in mortality is by the ratio of  $a/b$ . Pace-driven populations are characterized by relatively high  $a$  and relatively low  $b$  values; shape-driven populations are characterized by relatively high  $b$  and relatively low  $a$  values. The ratio  $a/b$  scales sensitivities of entropy (see (26) and (27)) and its magnitude can be interpreted as the relative importance of pace versus shape in a population.

Although there was little consistent change in  $a/b$  in the 19<sup>th</sup> century (Appendix, Figure A-1), Figure 2 reveals two separate trends over the first and the second half of the 20<sup>th</sup> century. We observe a steep decline until 1950 followed by a much slower decline thereafter. The shift in dynamics around 1950 is consistent with previous findings on

the rectangularization and shift of the survival curve (Ebeling, Rau, and Baudisch 2018; Cheung et al. 2005; Canudas-Romo 2008; Bergeron-Boucher, Ebeling, and Canudas-Romo 2015; Bongaarts 2005; Kannisto 1996). The regular change in  $a/b$  over time, together with the kink in the trend, suggests that this ratio could be used to signal further trend changes in the future.

**Figure 2:** The ratio  $a/b$  for selected countries, zoomed into values below 0.1, which are predominantly observed after 1900. See Appendix for the complete range

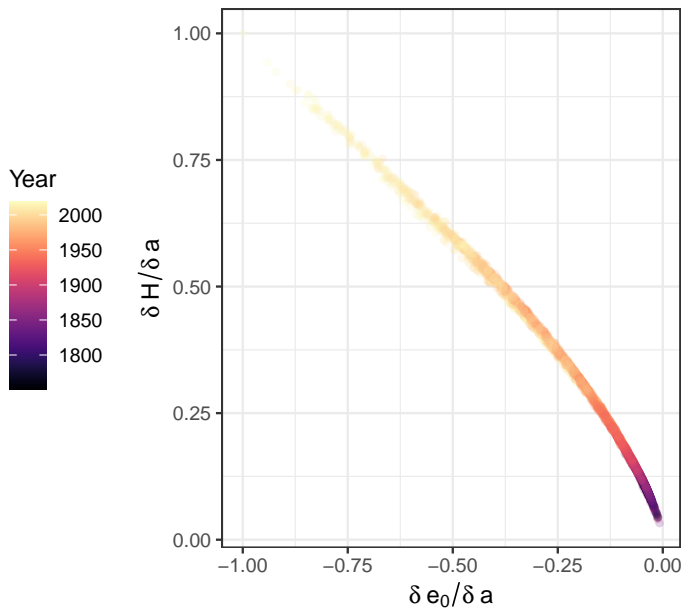


**Search for novel macro-level regularities.** Demographers aim to describe, model, and predict change in mortality and life expectancy in the past, present, and future. Macro-level summary measures, such as  $e_0$ ,  $e^\dagger$ , or  $\mathcal{H}$ , strongly aid such analysis. Remarkable regularities have been revealed within pace–shape space for human and primate populations on how longevity emerged over time (Aburto et al. 2020; Colchero et al. 2021, 2016). As life expectancy increases, life disparity tends to decline, although exceptions to this relationship exist (Aburto and van Raalte 2018; Aburto et al. 2020). Regularities of change are important information for developing forecasting methods (Pascariu, Canudas-Romo, and Vaupel 2018; Bergeron-Boucher et al. 2018; Torri and Vaupel 2012). The relationships presented here, in particular those for  $e_0$  and  $\mathcal{H}$  as pace and shape measures respectively, can become instrumental for discovering new regularities.

Aburto et al. (2020) and Colchero et al. (2021, 2016) find strong regularities in pace–shape space – that is, for trends in life expectancy versus (scaled) lifetable entropy. In tight connection, Figure 3 shows the sensitivity of life expectancy and lifetable entropy to changes in the baseline level of mortality  $a$ . Similar to the linear relationships discovered in pace–shape space, the sensitivities of pace and shape with respect to the level of

mortality also follow a tight and almost linear trend. Notably, the analogous sensitivities of pace and shape with respect to the rate of aging (Figure A-2, Appendix) do not follow a similarly tight relationship.

**Figure 3:** Derivative of  $e_0$  with respect to  $a$  (calculated based on Equation 8) by derivative of  $\mathcal{H}$  with respect to  $a$  (calculated based on Equation 17)



Finding a strong regularity for  $a$  but not for  $b$  is consistent with results by Colchero et al. (2021). They demonstrate that their tight relationship between life-expectancy and life-table entropy results from changes in  $a$  but not in  $b$ . Together with their finding, the relationship in Figure 3 may further aid understanding of changes in the life-table entropy given changes in baseline mortality and their effect on life expectancy.

We emphasize that the results in the figures presented here do not include the effect of juvenile mortality and do not account for death beyond age 90. We also do not necessarily capture mortality by the best fitting model at every time point, and our period-based analysis is limited by the corresponding assumption of a hypothetical cohort, but we argue that relying on a Gompertz pattern fitted to real-world data provides a reasonable view of the ongoing dynamics that can lead to helpful general insights.

## **5. Conclusion**

We conclude that our central relationship and those derived from it hold promising potential for discovering and modeling macro-level regularities in lifespan and lifespan disparity, which may aid development of novel forecasting methods. It could further aid comparative research across species to quantify the relative impact of the environment on species' life histories. Last but not least, the analysis presented here exemplifies the beauty and power of formal demography to capture the essence of mortality change – or in Jim Vaupel's words – “the champagne of demography” and “the stories that angels tell each other on Sundays.” We dedicate this piece to him.

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## A. Appendix

### A.1 Changes in life disparity

#### A.1.1 Changes in life disparity with respect to $a$

Taking the derivative of Equation (7) with respect to  $a$  gives

$$(29) \quad \frac{de^\dagger}{da} = 0 - \frac{e_0}{b} - \frac{a}{b} \frac{de_0}{da}.$$

Inserting (8) gives

$$(30) \quad \frac{de^\dagger}{da} = -\frac{e_0}{b} + \frac{a}{b} \frac{e^\dagger}{a},$$

which simplifies to

$$(31) \quad \frac{de^\dagger}{da} = \frac{1}{b} (e^\dagger - e_0)$$

and proves Equation (10).

#### A.1.2 Changes in life disparity with respect to $b$

Taking the derivative of equation (7) with respect to  $b$  gives

$$(32) \quad \frac{de^\dagger}{db} = -\frac{1}{b^2} + \frac{ae_0}{b^2} - \frac{a}{b} \frac{de_0}{db}$$

Taking out the inverse rate of aging such that

$$(33) \quad \frac{de^\dagger}{db} = \frac{1}{b} \left( -\frac{1}{b} (1 - ae_0) - a \frac{de_0}{db} \right)$$

and setting  $1 = e_0/e_0$  within the internal brackets, the first term changes into

$$(34) \quad \frac{de^\dagger}{db} = \frac{1}{b} \left( -\frac{e_0}{b} \left( \frac{1}{e_0} - a \right) - a \frac{de_0}{db} \right).$$

Replacing for equation (16) in the internal brackets

$$(35) \quad \frac{de^\dagger}{db} = \frac{1}{b} \left( -e_0 \mathcal{H} - a \frac{de_0}{db} \right).$$

and remembering that entropy  $\mathcal{H}$  is the ratio of  $e^\dagger/e_0$ , this simplifies to

$$(36) \quad \frac{de^\dagger}{db} = \frac{1}{b} \left( -e^\dagger - \frac{de_0}{db} \right).$$

Noting (8) and taking out  $a$  finally leads to

$$(37) \quad \frac{de^\dagger}{db} = \frac{a}{b} \left( \frac{de_0}{da} - \frac{de_0}{db} \right)$$

which proves (11).

## A.2 Changes in entropy

Taking the derivative of entropy (16) with respect to  $a$

$$(38) \quad \frac{d\mathcal{H}}{da} = \frac{1}{b} \left( -\frac{1}{e_0^2} \frac{de_0}{da} - 1 \right),$$

and inserting equation (9) for the derivative

$$(39) \quad \frac{d\mathcal{H}}{da} = \frac{1}{b} \left( \frac{1}{e_0^2} \frac{e^\dagger}{a} - 1 \right)$$

gives

$$(40) \quad \frac{d\mathcal{H}}{da} = \frac{1}{b} \left( \frac{\mathcal{H}}{ae_0} - 1 \right). \quad \square$$

Taking the derivative of entropy (16) with respect to  $b$  gives

$$(41) \quad \frac{d\mathcal{H}}{db} = -\frac{1}{b^2} \left( \frac{1}{e_0} - a \right) - \frac{1}{b} \frac{1}{e_0^2} \frac{de_0}{db}.$$

This can further be rearranged into

$$(42) \quad \frac{d\mathcal{H}}{db} = -\frac{1}{b^2 e_0} + \frac{a}{b^2} - \frac{1}{b^2 e_0^2} (e_{\dagger} - e_0),$$

and by moving the second term to the end and accounting for (16) into

$$(43) \quad \frac{d\mathcal{H}}{db} = -\frac{1}{b^2 e_0} (1 + (\mathcal{H} - 1)) + \frac{a}{b^2},$$

which reduces to

$$(44) \quad \frac{d\mathcal{H}}{db} = \frac{1}{b^2} \left( a - \frac{\mathcal{H}}{e_0} \right). \quad \square$$

### A.3 Relative changes

Multiplying (8) by  $a/e_0$  and (9) by  $b/e_0$  leads respectively to the scaled relative changes

$$(45) \quad \frac{a}{e_0} \frac{de_0}{da} = -\frac{a}{e_0} \frac{e^{\dagger}}{a} = -\mathcal{H}$$

and

$$(46) \quad \frac{b}{e_0} \frac{de_0}{db} = \frac{b}{e_0} \frac{1}{b} (e^{\dagger} - e_0) = \mathcal{H} - 1.$$

Summing both equations proves Equation (12).

Similarly, multiplying (10) by  $a/e^\dagger$  and (11) by  $b/e^\dagger$  leads respectively to the scaled relative changes

$$(47) \quad \frac{a}{e^\dagger} \frac{de^\dagger}{da} = \frac{a}{e^\dagger} \frac{1}{b} (e^\dagger - e_0) = \frac{a}{b} \left( 1 - \frac{1}{\mathcal{H}} \right)$$

and

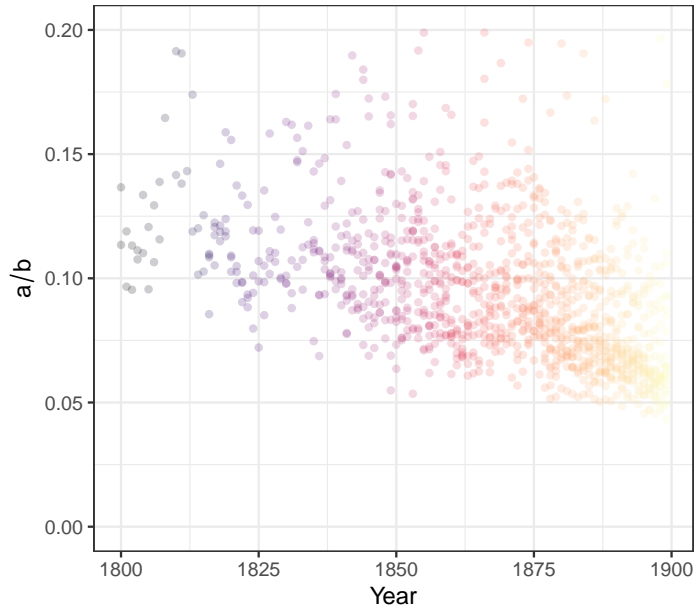
$$(48) \quad \frac{b}{e^\dagger} \frac{de^\dagger}{db} = \frac{b}{e^\dagger} \frac{a}{b} \left( \frac{de_0}{da} - \frac{de_0}{db} \right) = \frac{b}{e^\dagger} \frac{a}{b} \left( -\frac{e^\dagger}{a} - \frac{1}{b} (e^\dagger - e_0) \right) = \\ -1 - \frac{a}{b} \left( 1 - \frac{1}{\mathcal{H}} \right).$$

Summing both equations proves Equation (13).



#### A.4 Further figures

**Figure A-1: The ratio  $a/b$  for selected countries before 1900, full range of observed values**



**Figure A-2: Derivative of  $e_0$  with respect to  $b$  by derivative of  $\mathcal{H}$  with respect to  $b$**

